stability relative to this special class. Moreover the problem can be extended, within the framework of our general method, by considering obstacles whose contours are composed of a finite number of analytic arcs with shock waves originating at their points of intersection.

It is evident that instability at V, or local instability, is sufficient to insure instability in the large. Hence, the above result on instability gives the complete answer to the problem of determining the conditions for instability of shock lines attached to the vertex V of an obstacle whose contour is an analytic curve. Since at most two shock angles α at V are mathematically possible the shock line which actually occurs and which corresponds to the shock line experimentally observed must therefore be the one whose inclination α lies in the interval $\alpha_0(M) < \alpha < \beta(M)$. This may be accepted as sufficient evidence for the stability (local or in the large) of shock lines with inclination α in the interval $\alpha_0(M) < \alpha < \beta(M)$ by those not interested in an existence-theoretic treatment of the problem.

[1] Prepared under Navy Contract N6onr-180, Task Order V, with Indiana University.

[2] The derivation of these relations and other results mentioned in this note are contained in several papers which we expect to publish later in the *Journal of Mathematics and Physics* under the following titles: "Calculation of the Curvatures of Attached Shock Waves"; "The Consistency Relations for Shock Waves"; and "The Distribution of Singular Shock Directions."

[3] This conclusion is reached by an observation of the graphs of the functions ω and $-G_0(M, \alpha)$ shown in the paper "Calculation of the Curvatures of Attached Shock Waves."

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*PROGRESS IN THE STATISTICAL THEORY OF TURBULENCE**

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The fundamental notion of statistical mean values in fluid mechanics was first introduced by Reynolds. His most important contributions were the definition of the mean values for the so-called Reynolds' stresses and the recognition of the analogy between the transfer of momentum, heat and matter in the turbulent motion.

In the decades following Reynolds' discoveries, the turbulence theory was directed toward finding semi-empirical laws for the mean motion by methods loaned from the kinetic theory of gases. Prandtl's ideas on momentum transfer and Taylor's suggestions concerning vorticity transfer belonged to the most important contributions of this period. I believe that my formulation of the problem by the application of the similarity principle has the merit to be more general and independent of the methods of the kinetic theory of gases. This theory led to the discovery of the logarithmic law of velocity distribution in shear motion for the case of homologous turbulence.

The next important step was the definition of isotropic turbulence by Taylor and the following period in the development of the theory of turbulence was devoted to the analysis of the quantities which are accessible to measurement in a wind tunnel stream. These quantities are the correlation functions and the spectral function. The general mathematical analysis of the correlations was executed by L. Howarth and myself. One has to consider five scalar functions $f(r)$, $g(r)$, $h(r)$, $k(r)$, $l(r)$. These functions determine all double and triple correlations between arbitrary velocity components observed at two points because of the tensorial character of the correlations. The two scalar functions for the double correlations are defined as follows:

$$f(r) = \frac{\overline{u_1(x_1, x_2, x_3)u_1(x_1 + r, x_2, x_3)}}{\overline{u_1^2}},$$

$$g(r) = \frac{\overline{u_1(x_1, x_2, x_3)u_1(x_1, x_2 + r, x_3)}}{\overline{u_1^2}}. \tag{1}$$

Because of the continuity equation for incompressible fluids $g = f + \frac{r}{2}\frac{df}{dr}$.

For the same reason the triple correlations h, k and l can be expressed by one of them, e.g., by

$$h(r) = \frac{\overline{[u_1(x_1, x_2, x_3)]^2 u_1(x_1 + r, x_2, x_3)}}{[\overline{u_1^2}]^{3/2}} \tag{1a}$$

In addition we also deduced a differential equation from the Stokes-Navier Equation which gives the relation between the time derivative of the function f and the triple correlation function h.

$$\frac{\partial}{\partial t}(f\overline{u^2}) + 2[\overline{u^2}]^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{2}$$

We discussed this equation in two special cases:

(*a*) Small Reynolds number—in this case the triple correlations can be neglected and one obtains a self-preserving form for the double correlation function as a function of r/λ, where λ is defined by the relation

$$\frac{d\overline{u^2}}{dt} = -10\nu\frac{\overline{u^2}}{\lambda^2}. \tag{3}$$

(*b*) Large Reynolds number—in this case the terms containing the viscosity can be neglected for not too small values of r and the functions f and h are assumed to be functions of the variable r/L; L is a length characterizing the scale of turbulence. The hypothesis of self-preserving correlation function leads to the following special results. One can consider three simple cases:

1. L = constant; then we have $\overline{u^2} \sim t^{-2}$ (Taylor).
2. Loitziansky has shown that if the integral $\overline{u^2}\int_0^\infty r^4 f(r)dr$ exists, it must be independent of time, consequently $\overline{u^2}L^5$ = constant. Then $\overline{u^2} \sim t^{-10/7}, L \sim t^{2/7}$.
3. If the self-preserving character is extended to all values of r, i.e., also near $r = 0$, one has $\overline{u^2} \sim t^{-1}, L \sim t^{1/2}$ (Dryden).

On the other hand, Taylor introduced a spectral function for the energy passing through a fixed cross-section of a turbulent stream as the Fourier transform $\mathfrak{F}_0(n)$ of the correlation function $f(r)$. The relation between $\mathfrak{F}$ and f is given by the following equations:

$$f(r) = \int_0^\infty \mathfrak{F}_0(n) \cos \frac{2\pi nr}{U} dn,$$
$$\mathfrak{F}_0(n) = \frac{4\overline{u^2}}{U} \int_0^\infty f(r) \cos \frac{2\pi nr}{U} dr. \tag{4}$$

In these equations n is the frequency of the fluctuation of the uniform velocity U as function of time. Relative to the stream, $\mathfrak{F}_0(n)$ can be replaced by $\mathfrak{F}_1(\kappa_1)$, where $\kappa_1 = \frac{2\pi n}{U}$, i.e., the wave number of the fluctuation, measured in the x_1 direction.

It is seen that in this period of the development of the turbulence theory the analytical and experimental means for the study of isotropic turbulence were clearly defined but (with the exception of the case of very small Reynolds numbers) no serious attempt was made to find the laws for the shapes of either the correlation or the spectral functions. I believe this is the principal aim of the period in which we find ourselves at present. Promising beginnings were made by Kolmogoroff, Onsager, Weizsäcker and Heisenberg. I do not want to follow the special arguments of these authors. I want rather to define the problem clearly and point out the relations between assumptions and results.

A—I will assume that the three components of the velocity in a homogeneous isotropic turbulent field, at any instant, can be developed in the manner of Fourier's integrals

$$u_i = \int\int_{-\infty}^{\infty}\int Z_i(\kappa_1, \kappa_2, \kappa_3, t)e^{i(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)} d\kappa_1 d\kappa_2 d\kappa_3. \tag{5}$$

B—The intensity of the turbulent field be characterized by the quadratic mean value $\overline{u_i^2}$ level of the turbulence). Also there exists a function $\mathfrak{F}(\kappa)$ such that $[\overline{u_i^2}]_0^\kappa = \int^\kappa \mathfrak{F}(\kappa')d\kappa'$, where the symbol $[\overline{u_i^2}]_0^\kappa$ means a partial mean value of the square of the velocity, the averaging process being restricted for such harmonic components whose wave numbers κ_1, κ_2, κ_3 satisfy the relation

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \leqq \kappa^2. \tag{6}$$

If such a function exists it is connected with the spectral function of Taylor $\mathfrak{F}_1(\kappa_1)$ by the relation

$$\mathfrak{F}_1(\kappa_1) = \frac{1}{4}\int_{\kappa}^{\infty} \frac{1}{\kappa^3}\mathfrak{F}(\kappa)(\kappa^2 - \kappa_1^2)d\kappa. \tag{7}$$

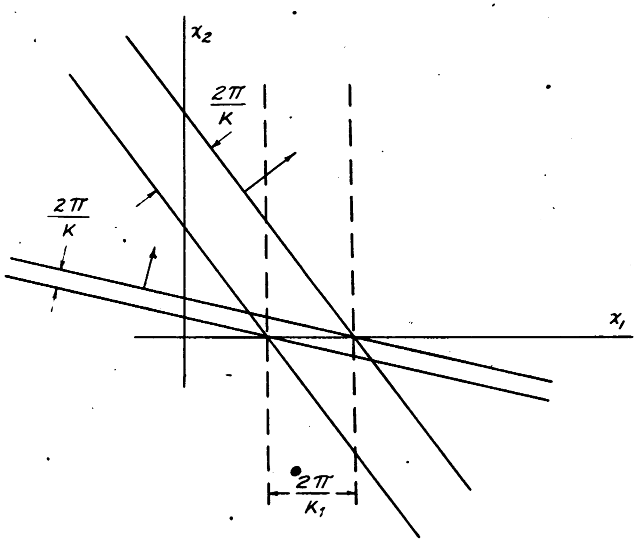

FIGURE 1

Contribution of oblique waves to plane waves in direction of x_1.

This relation was found by Heisenberg. It expresses the geometrical fact that all oblique waves, figure 1, whose wave-length $\frac{2\pi}{\kappa} < \frac{2\pi}{\kappa_1}$, necessarily contribute in the one-dimensional analysis to the waves with wave length $2\pi/\kappa_1$.

C—It is evident that there must be an equation for the time derivative of $\mathfrak{F}(\kappa)$ which corresponds to the equation for the time derivative of $f(r)$ which has been found by Howarth and myself. The physical meaning of this equation is evident. Let us start from the energy equation for a fluid element:

$$\frac{1}{2}\frac{\partial \overline{u_i^2}}{\partial t} + \overline{(u_i u_j + \delta_{ij} p/\rho)\frac{\partial u_i}{\partial x_j}} = \nu \overline{\frac{\partial^2 u_i}{\partial x_j^2} u_j}.$$

The right side represents the energy dissipation by viscous forces. The second term on the left side is the work of the Reynolds stresses; it represents a transfer of energy without actual dissipation. Our problem is to find $\partial\mathfrak{F}/\partial t$ by Fourier analysis and averaging process. One finds the contribution of the viscous forces to be equal to $-2\nu\mathfrak{F}(\kappa)\kappa^2$. Hence we write formally

$$\frac{\partial \mathfrak{F}}{\partial t} + \mathfrak{W}_\kappa = -2\nu\kappa^2\mathfrak{F}(\kappa). \tag{8}$$

Here $\mathfrak{W}_\kappa d\kappa$ is the balance for the energy contained in harmonic components comprised in the interval $d\kappa$; obviously $\int_0^\infty \mathfrak{W}_\kappa d\kappa = 0$. C. C. Lin has shown that $\mathfrak{W}_\kappa = 2\mathfrak{K}\kappa^2$, where

$$\mathfrak{K}(\kappa) = 2(\kappa^2\mathfrak{K}_1''(\kappa) - \kappa\mathfrak{K}_1'(\kappa))$$

and

$$\mathfrak{K}_1(\kappa) = \frac{2(\overline{u^2})^{3/2}}{\pi}\int^\infty h(r)\frac{\sin \kappa r}{\kappa}dr.$$

Unfortunately this relation does not help, as far as the determination of f and h is concerned. For example, if one expresses h in terms of f from the Kármán-Howarth equation, calculates $\mathfrak{W}_\kappa$ and substitutes the result in equation (8), one obtains an identity. It appears that at the present time one needs some additional physical assumption.

D—We assume that $\mathfrak{W}_\kappa$ can be expressed in the form:

$$\mathfrak{W}_\kappa = \int_0^\infty \Theta\{\mathfrak{F}(\kappa), \mathfrak{F}(\kappa'), \kappa, \kappa'\}d\kappa'. \tag{9}$$

The physical meaning of this assumption is the existence of a transition function for energy between the intervals $d\kappa$ and $d\kappa'$ which depends only on the energy density and the wave numbers of the two intervals. It follows from this definition that by interchanging κ and κ', one has

$$\Theta\{\mathfrak{F}(\kappa), \mathfrak{F}(\kappa'), \kappa, \kappa'\} = -\Theta\{\mathfrak{F}(\kappa'), \mathfrak{F}(\kappa), \kappa', \kappa\}. \tag{10}$$

It must be noted that our assumption probably cannot be exact. It is very probable that the values of $\mathfrak{F}$ for the difference and the sum of κ and κ' also enter in the transition function. I believe that the assumption gives a fair approximation when κ and κ' are very different, but it is certainly untrue if κ and κ' are nearly equal.

E—We furthermore specify the function Θ in the following way:

$$\Theta = -C\mathfrak{F}(\kappa)^{\alpha}\mathfrak{F}(\kappa')^{\alpha'}\kappa^{\beta}\kappa'^{\beta'}; \qquad C = \text{const.} \tag{11}$$

It follows from dimensional reasoning that

$$\alpha + \alpha' = {}^3/_2, \qquad \beta + \beta' = {}^1/_2.$$

As a result of the sequence of assumptions given above we obtain the equation:

$$\frac{\partial \mathfrak{F}}{\partial t} = C\left[\mathfrak{F}^{\alpha}\kappa^{\beta}\int_0^{\kappa} \mathfrak{F}(\kappa')^{3/2-\alpha}\kappa'^{1/2-\beta}d\kappa' - \mathfrak{F}^{3/2-\alpha}\kappa^{1/2-\beta}\int_{\kappa}^{\infty} \mathfrak{F}(\kappa')^{\alpha}\,\kappa'^{\beta}d\kappa'\right] - 2\nu\kappa^2\mathfrak{F}. \quad (12)$$

Obviously if $\mathfrak{F}(\kappa)$ is known for $t = 0$, equation (12) determines the values of $\mathfrak{F}$ for all times. If one neglects the first term on the left side, which represents the decay of turbulence and chooses the specific values $\alpha = {}^1/_2$, $\beta = -{}^3/_2$ one arrives to the theory proposed by Heisenberg.

Let us consider the case of large Reynolds number but assume that κ is not so large that the term containing the viscosity coefficient becomes significant. Let us also assume that the first term on the left side is small by comparison to the second term. Physically this means that the energy entering in the interval $d\kappa$ is equal to the energy which leaves the interval. Then one has the relation:

$$\mathfrak{F}^{\alpha}\kappa^{\beta}\int_0^{\kappa}\mathfrak{F}(\kappa')^{3/2-\alpha}\kappa'^{1/2-\beta}d\kappa' = \mathfrak{F}^{3/2-\alpha}\kappa^{1/2-\beta}\int_{\kappa}^{\infty}\mathfrak{F}(\kappa')^{\alpha}\kappa'^{\beta}d\kappa'. \quad (13)$$

This equation is satisfied by the solution $\mathfrak{F}(\kappa) \sim \kappa^{-5/3}$, as one easily can see by substitution in 13. This result is independent, evidently, of the special choice of α and β. That is the reason why it was independently found by Onsager, Kolmogoroff and Weizsäcker. It is essentially a consequence of dimensional considerations. Let us now stay with the case of large Reynolds numbers by neglecting again the viscosity term but retaining the first term on the left side. In other words we consider the actual process of decay at large Reynolds numbers. Let us assume that $\mathfrak{F}$ is a function of a non-dimensional variable κ/κ_0, when κ_0 is a function of time. This assumption is equivalent to our former assumption that $f(r)$ is a function of r/L; i.e., we assume that $\mathfrak{F}$ and f preserve their shapes during the decay. Evidently $\kappa_0 \sim 1/L$. Then the function $\mathfrak{F}$ can be written in the form

$$\mathfrak{F}(\kappa) = \frac{\overline{u^2}}{\kappa_0}\Phi\left(\frac{\kappa}{\kappa_0}\right).$$

Then with $\dfrac{\kappa}{\kappa_0} = \xi$ and

$$\frac{\partial}{\partial t}\mathfrak{F} = \frac{\Phi}{\kappa_0}\frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\Phi\frac{d\kappa_0}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\Phi'(\xi)\xi\frac{d\kappa_0}{dt}$$

equation (12) becomes

$$\left(\frac{1}{\kappa_0}\frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\frac{d\kappa_0}{dt}\right)\Phi - \frac{\overline{u^2}}{\kappa_0^2}\frac{d\kappa_0}{dt}\Phi'\xi + \mathfrak{W}_\kappa = 0, \tag{14}$$

where

$$\mathfrak{W}_\kappa = -C[\overline{u^2}]^{3/2}[\Phi^\alpha\xi^\beta \int_0^\xi \Phi(\xi')^{3/2-\alpha}\xi'^{1/2-\beta}d\xi' - \Phi^{3/2-\alpha}\xi^{1/2-\beta}\int_\xi^\infty \Phi(\xi')^\alpha \xi'^\beta d\xi'].$$

According to Loitziansky's results[1] $\dfrac{1}{\overline{u^2}}\dfrac{d\overline{u^2}}{dt} = \dfrac{5}{\kappa_0}\dfrac{d\kappa_0}{dt}$ and one obtains the equation

$$\xi^5(\xi^{-4}\Phi)' = -5C\frac{[\overline{u^2}]^{3/2}}{\dfrac{d\overline{u^2}}{dt}}[\Phi^\alpha\xi^\beta I_0^\xi - \Phi^{3/2-\alpha}\xi^{1/2-\beta}I_\xi^\infty], \tag{15}$$

where

$$I_0^\xi = \int_0^\xi \Phi(\xi')^{3/2-\alpha}\xi'^{1/2-\beta}d\xi'; \qquad I_\xi^\infty = \int_\xi^\infty \Phi(\xi')^\alpha \xi'^\beta d\xi'.$$

Let us assume that $4\alpha + \beta < {}^5/_2$ as, for example, in the case of Heisenberg. Then for small values of ξ the right side of equation (15) is small in comparison with the term on the left side and one has

$$\Phi(\xi) \cong \text{const. } \xi^4.$$

If $4\alpha + \beta > {}^5/_2$, $\mathfrak{F}$ begins with a lower power of κ than κ^4 and one can show that the integral $\int_0^\infty r^4 f(r)dr$ does not converge, so that Loitziansky's result is incorrect. I should like to investigate this second case in a later work. Let us assume, for the time being, that Loitziansky's result is correct and therefore the first case prevails. Then it follows that $\mathfrak{F}$ or Φ behaves as $(\kappa/\kappa_0)^4$ for small values of κ and is proportional to $(\kappa/\kappa_0)^{-5/3}$ for large values of κ. For any definite choice of α and β the differential equation (15) can be solved numerically. In June, 1947, I suggested to F. E. Marble that he carry out some such calculations and his results will be reported in a following publication. The result that $\mathfrak{F} \cong \kappa^4$ for small values of κ was also found in a different way by C. C. Lin.

For the time being I propose an interpolation formula as follows:

$$\Phi(\xi) = \text{const. } \frac{\xi^4}{(1+\xi^2)^{17/6}}. \tag{16}$$

This interpolation formula represents correctly $\Phi(\xi)$ for small and large values of ξ and has the advantage that all calculations can be carried out analytically by use of known functions. The results are as follows:

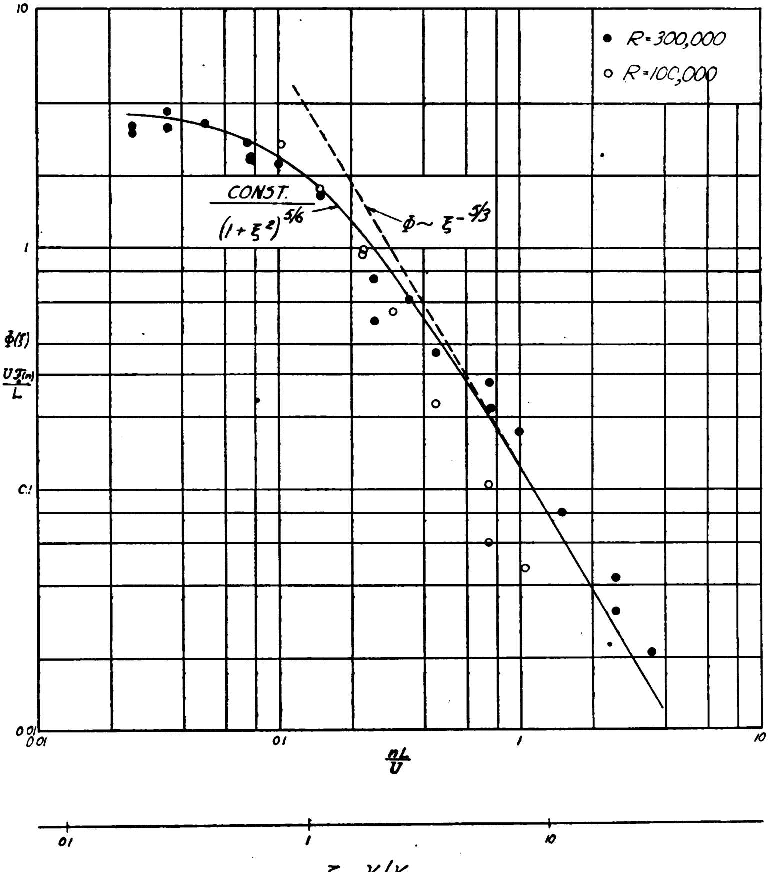

FIGURE 2

Comparison of observed and computed values of the frequency spectrum.

$$\mathfrak{F}(\kappa/\kappa_0) = \text{const.} \frac{(\kappa/\kappa_0)^4}{[1 + (\kappa/\kappa_0)^2]^{17/6}}$$

$$\mathfrak{F}_1(\kappa_1/\kappa_0) = \text{const.} \frac{1}{[1 + (\kappa_1/\kappa_0)^2]^{5/6}}$$

$$f(\kappa_0 r) = \frac{2^{2/3}}{\Gamma(1/3)} (\kappa_0 r)^{1/3} K_{1/3}(\kappa_0 r) \qquad (17)$$

$$g(\kappa_0 r) = \frac{2^2/_3}{\Gamma(1/3)} (\kappa_0 r)^{1/3} \left[K_{1/3}(\kappa_0 r) - \frac{\kappa_0 r}{2} K_{-2/3}(\kappa_0 r) \right].$$

The K's are Bessel functions with imaginary argument. For small values of $\kappa_0 r$

$$f(\kappa_0 r) = 1 - \frac{\Gamma(^2/_3)}{\Gamma(^4/_3)}\left(\frac{\kappa_0 r}{2}\right)^{2/3} \tag{18}$$

as suggested by Kolmogoroff's theory.

I have compared these results with the measurements of Liepmann Laufer and Liepmann[2] carried out at the California Institute of Technology with the financial assistance of the N.A.C.A.† These observations were made in the 10-foot wind tunnel of the Guggenheim Aeronautical Laboratory using a grid whose mesh size was $M = 4$ inches. The measurements were made at a distance $x = 40.4M$ from the grid. Figure 2 shows the comparison of calculated and measured values for the spectral function

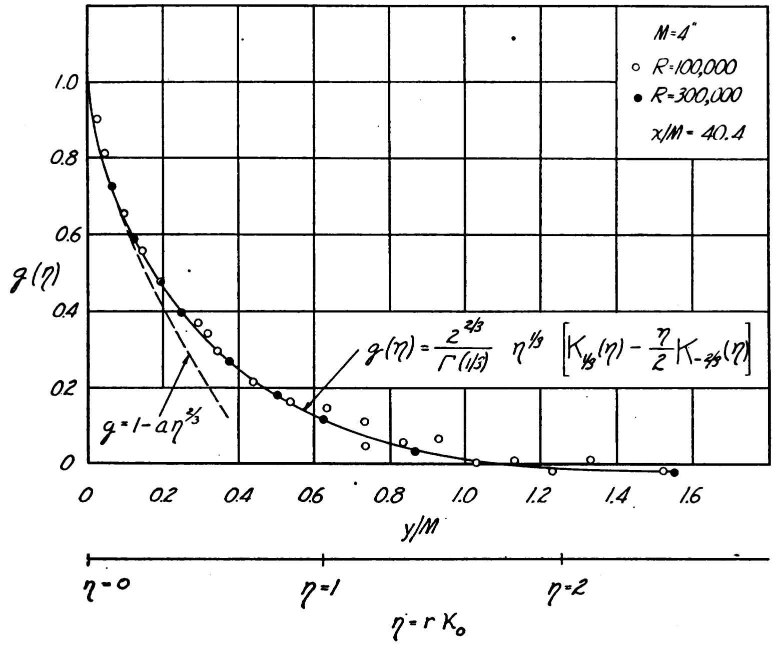

FIGURE 3

Comparison of observed and computed values of the correlation function g. The Reynolds number is based on the stream velocity and the mesh size.

$\mathfrak{F}_1(\kappa_1)$. It has to be taken into account that the observed values of $\mathfrak{F}_1(\kappa_1)$ have large scatter; the deviation for high values of κ_1 corresponds to the beginning influence of viscosity. Figure 3 gives a comparison between measured and calculated values of the correlation function $g(r)$. This function is chosen because the observations are more accurate than in any other case. It is seen that the agreement is almost too good in view of the assumptions made above. One must remark that there is only one arbitrary constant in the formula for g, viz., the constant κ_0 which determines the scale of the turbulence. It is true that some of the data of reference 2 do not show such a good agreement. The agreement is excellent for values of g larger than 0.1, but after that the measured values are higher than the calculated ones. Possibly some oscillations existing in the wind tunnel stream were interpreted as turbulence or the turbulence is not quite isotropic.

I believe that the merits of my deduction are: (a) the assumptions involved are exactly formulated; (b) the specific assumptions of Heisenberg's theory concerning the transition function are not used; (c) the actual process of decay is considered; (d) the analysis is extended to the lower end of the turbulence spectrum.

Concerning the case of large values of κ (small values of r) L. Kovásznay[3] introduced an interesting assumption which is more restricting than my assumption D. Obviously $\int_0^{\kappa} \mathcal{W}_\kappa d\kappa$ is the total energy transferred by the Reynolds stresses from the interval $(0 \rightarrow \kappa)$ to the interval $(\kappa \rightarrow \infty)$. Kovásznay assumes—following Kolmogoroff's arguments—that this quantity is a function of $\mathfrak{F}(\kappa)$ and κ only. Then for dimensional reasons $\int_0^{\kappa} \mathcal{W}_\kappa d\kappa = \text{const. } \mathfrak{F}^{3/2}\kappa^{5/2}$. This assumption appears to be correct for large values of κ. When, however, the assumption is extended to the range of small values of κ and one substitutes $\mathcal{W}_\kappa$ in equation (8) one can calculate easily $\mathfrak{F}(\kappa)$. Neglecting the viscous term, one obtains the relation

$$\mathfrak{F}(\kappa/\kappa_0) = \text{const. } \frac{(\kappa/\kappa_0)^4}{[1 + \mathfrak{F}^{1/2}(\kappa/\kappa_0)^{3/2}]^{17/3}} \tag{19}$$

The right side of equation (19) behaves as my corresponding equation (17) for small and large values of κ/κ_0. It will be interesting to see how far the different transition from small to large values influences the accordance with observation.

* Presented at the Heat Transfer and Fluid Mechanics Institute, Los Angeles, California, June 23, 1948.

† The N.A.C.A. has kindly allowed presentation of these data prior to official N.A.C.A. publication.

[1] Loitziansky, L. G., "Some Basic Laws of Isotropic Turbulent Flow," Central Aero-Hydrodynamical Institute, Report No. 440, Moscow, 1939. Translated as N.A.C.A. Technical Memorandum 1079.

[2] Liepmann, H. W., Laufer, J., and Liepmann, K., "On Some Turbulence Measurements Behind Grids," Final Report N.A.C.A. Contract NAw 5442, July, 1948.

[3] Kovásznay, Leslie, S. G., "The Spectrum of Locally Isotropic Turbulence," *Phys. Rev.*, **73**, (9), 1115 (May, 1948).